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Center Concepts on Distance *k*-Dominating Sets

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Abstract: A set $\boldsymbol{D} \subseteq \boldsymbol{V}(\boldsymbol{G})$ is called a *dominating set* of **G** if every vertex in $\boldsymbol{V}(\boldsymbol{G}) - \boldsymbol{D}$ is adjacent to some vertex in $\boldsymbol{D}$. A set $\boldsymbol{D} \subseteq \boldsymbol{V}$ is called a distance *k*-dominating set of *G* if each $x \in V - D$ is within distance *k* from some vertex of *D*. In this paper, we determine the distance-$\boldsymbol{k}$ domination number for a given graph using the *k*-center and link vector concepts. Using the *k*-center concept we present some necessary and sufficient condition for distance-*k* dominating set.

Keywords: Distance, radius, domination number, distance *k*-domination number, *k*-center, reachable set, link vector.

1. INTRODUCTION

Let $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$ be a simple graph. The distance between $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted by $d(u, v)$, is the length of a shortest $\boldsymbol{u} - \boldsymbol{v}$ path. For a vertex $\boldsymbol{v} \in \boldsymbol{V}$ and a positive integer *k*, the *k*-neighborhood of $\boldsymbol{v}$ in $\boldsymbol{G}$ is defined as $\boldsymbol{N_k}(\boldsymbol{v})$ $=\{\boldsymbol{u} \in \boldsymbol{V}(\boldsymbol{G}) / \boldsymbol{d}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{k}\}$. For $\boldsymbol{k} = \boldsymbol{1}$, $\boldsymbol{N_1}(\boldsymbol{v})$ is the neighborhood of $\boldsymbol{v}$ and simply denoted by $\boldsymbol{N}(\boldsymbol{v})$. Let $\boldsymbol{d}(\boldsymbol{x}) = |\boldsymbol{N}(\boldsymbol{x})|$ be the degree of *G* and $\boldsymbol{\delta}$ and Δ be the minimum and maximum degree of $\boldsymbol{G}$, respectively. The set $\boldsymbol{N_k}[\boldsymbol{v}] = \boldsymbol{N_k}(\boldsymbol{v}) \cup \{\boldsymbol{v}\}$ is called the closed $\boldsymbol{k}$ neighborhood $\boldsymbol{v}$ in $\boldsymbol{G}$.

For a connected graph G, the eccentricity $e(v) = \max\{d(u, v) : \forall u \in V(G)\}$ and the eccentric set $E(v) = \{ u \in V : d(u, v) = e(v)\}$. The minimum eccentricity among the vertices of G is its radius and the maximum eccentricity is its *diameter*, which is denoted by $rad(G)$ and $diam(G)$, respectively. A vertex v in G is a central vertex if $e(v) = rad(G)$ and the subgraph induced by the central vertices of G is the *center* $Cen(G)$ of G. In this paper, we present the relation between distance-*k* dominating set and *k*-center of the given graph. We study the binary operations $\vee$, $\wedge$ in [1]. Using these operations we construct algorithm to find the distance *k*-dominating set.

Definition 1.1: A set $D \subseteq V(G)$ is called a *dominating set* of G if every vertex in $V - D$ is adjacent to some vertex in D.

The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. We call the set of vertices as a γ-set if it is a dominating set with cardinality $\gamma(G)$.

Definition 1.2: A set $D \subseteq V(G)$ is called a *distance k-dominating set* of G if $N_k[D] = V$. The *distance k- domination number* $\gamma_k(G)$ of G.

2. k-center [1]

Definition 2.1: Let S be a subset of V with k vertices. Let $v \in V$. Then the distance of S from v is defined as $d(S, v) = min\{ d(x, v) \ / \ x \in S\}$. If $v \in S$ then $d(S, v) = 0$. The eccentricity of S is the maximum of $d(S, v)$ over all $v \in V$. That is, $e(S) = max\{ d(S, v) \ / \ v \in V\}$.Consider the family F_k of the subset S of k vertices ($1 \leq k \leq n$-1) of G. The k-*center* of the graph G is the set S^* of k vertices of G such that, $e(S^*) = Min\{e(S), S \in F_k \}$. This minimum eccentricity is called the *radius of k-center* and it is denoted by $r_k(G)$.

Theorem 2.2

Every central vertex with radius k forms a distance k-dominating set.

Proof

Let G be a graph with radius k. Let $C(G)$ be the center of the graph G. Let $C(G) = \{ v_1, v_2, \ldots, v_m \}$. If $v_i \in C(G)$, $(1 \leq i \leq m)$, then $e(v_i) = k$. Hence $d(v_i, v) = k$ for some *v* in *V*. Therefore $d(v_i, v) \leq k$ for all $v \in V$ i.e, Each v_i is with distance k to all other vertices in G. Hence each v_i can dominate all the vertices of *G* with distance *k*. Hence every center vertex with radius k forms a distance k-dominating set. □

Theorem 2.3

For any connected graph with radius k, $\gamma_k(G) = 1$.

Proof

Let G be any connected graph. Let k be the radius of G. Let *C*(*G*) be the center of *G*. Then $e(v) = k$ for all $v \in C(G)$. Let $v \in C(G)$ then $max\{d(v, u) = k; u \in V\}$. Also we have $d(v, u) \leq k; u \in V$. Hence *v* dominates every vertex within distance k. So $\gamma_k(G) = 1$. □

Theorem 2.4

For any connected graph *G*, $\gamma_k(G) = 1$ if and only if there exists a vertex in G with eccentricity$\leq k$.

Proof

Let G be any connected graph and $V(G) = \{v_1, v_2, \ldots, v_n\}$. Suppose that $\gamma_k(G) = 1$. Let D be a minimum distance k-dominating set. Let $x_i \in D$.

Case (i): If $r(G) \leq k \leq diam(G)$, then there exists a vertex can dominate all the vertices within distance *k*. So that $d(v_i, v) \leq k \ \forall \ v \in V - \{v_i\}$. Which implies that $e(v_i) = max\{d(v_i, v), v \in V - \{v_i\}\} = k$.

Case(ii): If $k > diam(G)$, then there exists a vertex v_i can dominate all other vertices of *G* with distance less than *k*. Hence $d(v_i, v) < k \ \forall \ v \in V - \{v_i\}$. Which implies that $e(v_i) = max\{d(v_i, v), v \in V - \{v_i\}\} < k$.

Conversely, assume that there exists a vertex with eccentricity$\leq$ k. Let v_i be a vertex with $e(v_i) \leq k$. Then obviously, $d(v_i, v) \leq k$ $\forall$ $v \in V$-$\{v_i\}$. Hence $D = \{v_i\}$ can dominate all other vertices of G. Then D is a minimum distance k-dominating set of G. Hence $\gamma_k(G) = 1$. □

Theorem 2.5

Every k-center of G with radius i is a distance i-dominating set.

Proof

Let G be any connected graph with n vertices. Let S_k be the k-center of G with radius i. Hence $|S_k| = k$ and $e(S_k)$= i. That is the distance of S_k from the farthest vertex is i. Therefore, S_k dominates the farthest vertex with distance i. Hence S_k dominates all vertices of V within distance i and so S_k is a distance i-dominating set. □

Definition 2.6: The set of all vertices of the graph G, from which the vertex x is connected within a minimum distance λ is called as a reachable set of x within a distance λ and is denoted as $R_\lambda(x) = \{ y \in V / d(y, x) \leq \lambda\}$. Call this distance λ as penetration.

Definition 2.7: Characterize each vertex as a n-tuple. Each place of n-tuple can be represented by a binary zero or one. Call this n-tuple as a link vector simply LV of a vertex.

Thus a link vector $(j_1 j_2, \ldots, j_n)$ represent a vertex x_j where $j_k = 1$ if x_k is reachable within the penetration λ from x_j and zero otherwise. Denote a link vector of the vertex x by x' and denote the set of all link vectors as Ω.

If all the coordinate of a link vector of a vertex are equal to one then the link vector is said to be full and is denoted as (1). If all the coordinates of a link vector of a vertex are equal to zero then the link vector is said to be null and it is denoted by (0).

Definition 2.8: Let G be a graph. Let Ω be the set of LVs of all vertices. Define two binary operations $\vee$(cup) and $\wedge$(cap) as follows:

$\vee, \wedge : \Omega \times \Omega \to \Omega$ by

$(a_1, a_2, \ldots, a_n) \vee (b_1, b_2, \ldots, b_n) = (c_1, c_2, \ldots, c_n)$

where $c_i = max\{a_i, b_i\}$ & i= 1 to n

$(a_1, a_2, \ldots, a_n) \wedge (b_1, b_2, \ldots, b_n) = (c_1, c_2, \ldots, c_n)$

where $c_i = min\{a_i, b_i\}$ & i = 1 to n

Theorem 2.9

Let G be a graph with n vertices and $r_k(G) = i$. Let $D \subseteq V$ of k vertices ($1 \leq k \leq n$-1). Then D is a distance i-dominating set if and only if D is a k-center.

Proof

Let $D \subseteq V$ be a set of k vertices with $r_k(G) = i$. Suppose that D is a distance i-dominating set. Then there exists a vertex v in D such that $d(u, v) \leq i$, for every $u \in V - D$. $\therefore e(D) = i = r(D)$. It implies that D is a k-center. Conversely, suppose that D is a k-center with radius i. By theorem 2.5, D is a distance i- dominating set.

Now we take i =1, then we have the following corollary. □

Corollary 2.10 [1]

In any graph G with radius 1, a set D of k vertices $1 \leq k \leq n - 1$ is a dominating set if and only if D is a k-center. □

Theorem 2.11

Let G be a graph with n vertices. Then $\bigvee_{j=1}^{k} x_j'$ is full for a least integer k in G for $\lambda = i$ if and only if $D = \{x_1, x_2, \ldots, x_k\}$ is a minimum distance i-dominating set.

Proof

Consider the amount of penetration $\lambda = i$. Suppose that $\bigvee_{j=1}^{k} x_j'$ is full where x_j' is the LV of x_j. Take $x_j' = (x_{j_1}, x_{j_2}, \ldots, x_{j_n})$ for a least integer k. Now$\bigvee_{j=1}^{k} x_j' = (x_{1_1}, x_{1_2}, \ldots, x_{1_n}) \vee (x_{2_1}, x_{2_2}, \ldots, x_{2_n}) \vee \ldots \vee (x_{k_1}, x_{k_2}, \ldots, x_{k_n})$. Since $\bigvee_{j=1}^{k} x_j'$ is full, then $(x_{1_1}, x_{1_2}, \ldots, x_{1_n}) \vee (x_{2_1}, x_{2_2}, \ldots, x_{2_n}) \vee \ldots \ldots \vee (x_{k_1}, x_{k_2}, \ldots, x_{k_n}) = (1, 1, \ldots .1)$. Hence $D = \{x_1, x_2, \ldots, x_k\}$ dominates V and it is a minimum distance i-dominating set. Since k is minimum.

Conversely, suppose that $D = \{x_1, x_2, \ldots, x_k\}$ is a minimum distance i-dominating set. Then a vertex not in D is adjacent to at least one vertex of D within distance $\lambda = i$. That is, $d(D, y) \leq i$ $\forall$ $y \in V$-D. Thus all coordinates of any one of $x_1', x_2', \ldots, x_k'$ is 1. Hence $x_1' \vee x_2' \vee \ldots \vee x_k'$ is full, that is $\bigvee_{j=1}^{k} x_j'$ is full. It completes the proof. □

Theorem 2.12

Let G be a graph with n vertices. Then there exists a vertex whose link vector is full with $\lambda = k$ if and only if $\gamma_k(G) = 1$.

Proof

Let G be a graph with n vertices. Suppose that there exists a vertex v_i whose link vector is full with penetration k. That is, the j^{th} coordinate of $v_i'$is 1 for every j ($1 \leq j \leq n$). Hence the vertex v_i is reachable to all other vertices of G with penetration k. Hence this vertex v_i alone forms a distance k dominating set. Hence D is a minimum distance k -dominating set and so $\gamma_k(G) = 1$. Conversely, assume that $\gamma_k(G) = 1$. Let D be a γ_k-set of G. Take $D = \{v_i\}$. Then the vertex v_i dominates all other vertices within distance k. Hence v_i is reachable to all other vertices of G with $\lambda = k$. Then the LV v_i' of v_i is full. □

Theorem 2.13

If $r(G) \leq k \leq diam(G)$, then there exist a LV x_i' which is full with $\lambda = k$.

Proof

Let G be a graph with n vertices. Assume that $r(G) \leq k \leq diam(G)$. Then there exists a vertex x_i of G with eccentricity k. By theorem 2.4, $\gamma_k(G) = 1$. Then by theorem 2.12, x_i' is full with $\lambda = k$. □

Theorem 2.14

Let G be a connected graph with n vertices. Then $\wedge_{i=1}^{n} x_i'$ is full with $\lambda = 1$ if and only if G is complete.

Proof

Let G be a connected graph with n vertices. Suppose that $\wedge_{i=1}^{n} x_i'$ is full with $\lambda = 1$.

Then $\wedge_{i=1}^{n} x_i' = min[(x_{1_1}, x_{1_2}, \dots, x_{1_n}) \wedge (x_{2_1}, x_{2_2}, \dots, x_{2_n}) \wedge \dots \wedge (x_{n_1}, x_{n_2}, \dots, x_{n_n})] = (1, 1, \dots, 1)$. Since the j^{th} coordinate of x_i is full for all i, j $(1 \leq i, j \leq n)$, then the vertex x_i is reachable to all other vertices. Hence G is complete.

Conversely, Take $\wedge_{i=1}^{n} x_i' = min[(x_{1_1}, x_{1_2}, \dots, x_{1_n}) \wedge (x_{2_1}, x_{2_2}, \dots, x_{2_n}) \wedge \dots \wedge (x_{n_1}, x_{n_2}, \dots, x_{n_n})]$. Since G is complete, link vector of every vertex is full. Hence $\wedge_{i=1}^{n} x_i'$ is full. Thus the link vector concept is very useful to prove many results. □

Algorithm 2.15

Algorithm to find a minimum distance i-dominating set

Input. A graph $G = (V, E)$ with $V(G) = \{x_1, x_2, \dots, x_n\}$ with distance matrix and $diam(G) = d$. Find all reachable sets $R_\lambda(x_j)$ of x_j and find the link vector x_j' of x_j

Output. Minimum distance i-dominating set.

Step 1. $i = 1$ to d

Step 2.

2.1. Take w' is the LV of w. Initialize $w' \leftarrow (0)$ and $D = \emptyset$

2.2. For $j = 1$ to n

$w' = w' \vee x_j'$

$D = D \cup \{x_j\}$

2.3. If w' is not full then go to step 3.2

2.4. Print D is a minimum distance i -dominating set.

Otherwise go to step 3.2

2.5 Go to step 1.

This algorithm finds d number of minimum distance i-dominating sets, for $i = 1$ to d. It works with a for loop $j = 1$ to n, for a fixed i. To find a minimum distance i-dominating set this algorithm works with the operations $\vee$ and $\cup$ in 2n times. Totally it works in $2nd$ times and so it is a polynomial time algorithm. □

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