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Memory Effect on Decay Property of Solutions to Plate Equations

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Abstract: In this paper we focus on the initial-value problem of linear plate equations with memory in multi-dimensions, the decay structure of which is of regularity-loss property. We obtain fundamental solutions by using Fourier transform and Laplace transform. By virtue of the point-wise estimate of solutions in the Fourier space, we gain estimates and properties of solution operators, by utilizing which decay estimates of solutions to the linear problem are obtained and the decay rate can be as large as desired if the initial data are sufficiently smooth.

Keywords: plate equation; memory; decay estimates; regularity-loss type; initial-value problem

1. INTRODUCTION

In this paper we consider the initial-value problem of the following linear plate equation with memory term in multi-dimensional space R^n ($n \geq 1$):

$$u_{tt} + (1+\Delta^2)u - g * u = 0, \qquad (1.1)$$

with the initial data

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x). \qquad (1.2)$$

Here $u = u(x,t)$ is the unknown function of $x = (x_1,...,x_n) \in R^n$ and $t > 0$, and represents the transversal displacement of the plate at the point x and the time t. The term $-g * u = -\int_0^t g(t-\tau)u(\tau)d\tau$ accords with the memory term which reflects that the stress at an instant relies on the whole history of the strains the material has suffered, and g satisfies:

Assumption[A]:

a) $g \in C^2(R^+) \cap W^{2,1}(R^+)$,

b) $g(s) > 0, -C_0 g(s) \leq g'(s) \leq -C_1 g(s),$
$|g''(s)| \leq C_2 g(s), \forall s \in R^+,$

c) $1 - \int_0^t g(s)ds \geq C_3, \forall t \in R^+,$

where $C_i (i = 1,2,3)$ are positive constants.

In [8], Liu and S. Kawashima learned the decay property of a semi-linear plate equation with memory-type dissipation, whose linear part is given by:

$$u_{tt} + \Delta^2 u + u + g * \Delta u = 0, \qquad (1.3)$$

here the dissipation is given by the memory term $g * \Delta u$. In that paper, the authors obtained the global existence and the optimal decay estimates of solutions by introducing a set of time-weighted Sobolev spaces and using the contraction mapping theorem. They also showed that the dissipative structure is characterized by the function

$$\rho_1(\xi) = \frac{|\xi|^2}{1+|\xi|^4},$$

here $\rho_1(\xi)$ is introduced in the point-wise estimate in the Fourier space of solutions to the corresponding linear problem. $\rho_1(\xi)$ decides that the energy restricted in the either lower-frequency or higher-frequency domains decays polynomially and the decay property is of regularity-loss type

In [13], Liu and W. Wang studied the point-wise estimate of solutions to a dissipative wave equation

$$u_{tt} - \Delta u + u_t = 0, \qquad (1.4)$$

and they showed that the dissipative structure in (1.4) is characterized by the function $\rho_2(\xi) = \frac{|\xi|^2}{1+|\xi|^2}$.

This $\rho_2(\xi)$ determines that the energy restricted in the lower-frequency domain decays polynomially and exponentially in the higher-frequency domain and the decay property is of standard type instead of regularity-loss type. For more studies of such decay structure, we refer to [5, 6, 7, 15, 16, 17].

To have a better comparison of the dissipative structures, we study the equation (1.1).
Same as the above memory plate equation (1.3), the plate equation (1.1) is also of regularity-loss property. The decay structure of the regularity-loss type is characterized by the property $\rho(\xi) = \frac{1}{1+|\xi|^4}$, where $\rho(\xi)$ is introduced in the point-wise estimate in the Fourier space (3.1) of solutions to the linear problem. It is obvious that the decay structure is very weak in the higher-frequency domain since $\rho(\xi) \to 0$ as $\xi \to \infty$. In fact $\rho(\xi)$ determines that the energy restricted in the lower-frequency domain decays

exponentially and polynomially in the higher-frequency domain and the decay property is of regularity-loss type. There is one point worthy to be mentioned. The solutions in [8], [13] and this paper all decay polynomially. However, the dissipative structures are different. The decay rates in [8] and [13] are fixed, while the decay rate in this paper can be as large as desired if only the initial data are sufficiently smooth. For more studies on aspects of dissipation of plate equations, we refer to [1, 2, 9, 10]. Also, as for the study of decay properties for hyperbolic systems of memory-type dissipation, we refer to [3, 4, 11, 12, 14].

The prime objective of this paper is to study the decay estimates of solution to the initial-value problem (1.1)-(1.2). For our problem, due to the existence of memory term, it is a difficult task to obtain precisely the solution operator or its Fourier transform. While, by using Fourier transform and Laplace transform, we obtain the solution u to the linear problem (1.1)-(1.2) given by (2.4) and the solution operators $G(t)*$ and $H(t)*$. Furthermore, by employing the energy method in the Fourier space, we gain the point-wise estimate in the Fourier space of solutions to the problem (1.1)-(1.2). Appealing to this point-wise estimate, we obtain the point-wise of solution operators and their properties. Therefore, the decay estimates of solutions to (1.1)-(1.2) are achieved.

The contents of the paper are as follows. Solution formula are obtained in section 2. In section 3, we obtain the estimates and properties of solutions operators, which is based on the point-wise estimate in the Fourier space of solutions to the linear problem. In the last section, we prove the decay estimates of solutions to the linear problem by virtue of the properties of solution operators.

Before the end of this section, we give some notations to be used below. Let the Fourier transform of f indicated as $\mathcal{F}[f]$:

$$\mathcal{F}[f](\xi)=\hat{f}(\xi):=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{R^n}e^{-ix\cdot\xi}f(x)dx,$$

and we denote its inverse transform as $\mathcal{F}^{-1}$.

Let the Laplace transform of f indicated as $\mathcal{L}[f]$:

$$\mathcal{L}[f](\lambda):=\int_0^\infty e^{-\lambda t}f(t)dt,$$

and we denote its inverse transform as $\mathcal{L}^{-1}$.

$L^p=L^p(R^n)(1\le p\le\infty)$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$.

Z_+ denotes the totality of all the non-negative integers.

$W^{m,p}(R^n)$, $m\in Z_+$, $p\in[1,\infty]$ denote the usual Sobolev space with its norm

$$\|f\|_{W^{m,p}}:=(\sum_{k=0}^m\|\partial_x^k f\|_{L^p}^p)^{\frac{1}{p}}.$$

In particular, we use $W^{m,2}=H^m$. Here, for a nonnegative integer k , ∂_x^k denotes the totality or each of all the k -th order derivatives with resect to $x\in R^n$. Also, $C^k(I;H^m(R^n))$ denotes the space of k -times continuously differentiable functions on the interval I with values in the Sobolve space $H^m=H^m(R^n)$.

Finally, in this paper, we denote every positive constant by the same symbol C or c without confusion. $[\cdot]$ is Gauss' symbol.

2. Solution formula

In this section, our purpose is to obtain the solution formula of the problems (1.1)-(1.2). Suppose $G(x,t)$ and $H(x,t)$ are solutions to the following problems,

$$\begin{cases} G_{tt}+(1+\Delta^2)G-g*G=0, \\ G(x,0)=\delta(x), \\ G_t(x,0)=0. \end{cases} \tag{2.1}$$

$$\begin{cases} H_{tt}+(1+\Delta^2)H-g*H=0, \\ H(x,0)=0, \\ H_t(x,0)=\delta(x). \end{cases} \tag{2.2}$$

Apply Fourier transform and Laplace transform to (2.1) and （2.2）, then we have formally that

$$\hat{G}(\xi,t)=C\mathcal{L}^{-1}[\frac{\lambda}{1+|\xi|^4+\lambda^2-\mathcal{L}[g](\lambda)}](\xi,t),$$

$$\hat{H}(\xi,t)=C\mathcal{L}^{-1}[\frac{1}{1+|\xi|^4+\lambda^2-\mathcal{L}[g](\lambda)}](\xi,t).$$

here C is the constant determined by the initial data in (2.1) and (2.2).

Now we just compute $\hat{G}(\xi,t)$, similarly we could get $\hat{H}(\xi,t)$. First, apply Fourier transform to (2.1), we can obtain the following equation:

$$\begin{cases} \hat{G}_{tt}+(1+|\xi|^4)\hat{G}-g*\hat{G}=0, \\ \hat{G}(\xi,0)=\hat{\delta}(\xi), \\ \hat{G}_t(\xi,0)=0. \end{cases}$$

then apply Laplace transform to above equation, we can get

$$\int_0^\infty\hat{G}_{tt}e^{-\lambda t}dt+(1+|\xi|^4)\int_0^\infty\hat{G}e^{-\lambda t}dt -\int_0^\infty(g*\hat{G})e^{-\lambda t}dt=0,$$

by computing, we have that

$$-\lambda C+(1+|\xi|^4+\lambda^2)\mathcal{L}[\hat{G}](\lambda) -\mathcal{L}[g](\lambda)\cdot\mathcal{L}[\hat{G}](\lambda)=0,$$

so

$$\mathrm{L}[\hat{G}](\lambda)=\frac{C\lambda}{1+|\ \xi|^4+\lambda^2-\mathrm{L}[g](\lambda)},$$

finally, we have formally that

$$\hat{G}(\xi,t)=C\mathrm{L}^{-1}[\frac{\lambda}{1+|\ \xi|^4+\lambda^2-\mathrm{L}[g](\lambda)}](\xi,t).$$

Similarly,

$$\hat{H}(\xi,t)=C\mathrm{L}^{-1}[\frac{1}{1+|\ \xi|^4+\lambda^2-\mathrm{L}[g](\lambda)}](\xi,t),$$

here C is a constant determined by the initial data in (2.1).

Lemma2.1

$\hat{G}(\xi,t)$ and $\hat{H}(\xi,t)$ exist.

Proof.

We only prove $\hat{G}(\xi,t)$ exists; similarly we could prove $\hat{H}(\xi,t)$ exists.

Denote $F(\lambda):=\lambda^2+1+|\xi|^4-\mathrm{L}[g](\lambda)$.

To prove $\mathrm{L}^{-1}[\frac{\lambda}{F(\lambda)}]$ exists, we need to consider the zero points of $F(\lambda)$. Denote $\lambda=\sigma+i\nu$, $\sigma>-C_1$, C_1 is same as that in Assumption [A] b), then $\mathrm{L}[g](\lambda)$ exists. Assume that $\lambda_1=\sigma_1+i\nu_1$ is a zero point of $F(\lambda)$ and $\sigma_1>-C_1$, then σ_1 and ν_1 satisfy

$$\begin{cases}\mathrm{Re}F(\lambda_1) & =\sigma_1^2-\nu_1^2+1+|\xi|^4-\int_0^\infty\cos(\nu_1 t)e^{-\sigma_1 t}g(t)dt=0,\\ \mathrm{Im}F(\lambda_1) & =\int_0^\infty\sin(\nu_1 t)e^{-\sigma_1 t}g(t)dt+2\sigma_1\nu_1=0.\end{cases}\quad(2.3)$$

We claim that $\sigma_1\le 0$. Now we prove the claim by contradiction.

Assume that $\sigma_1>0$. If $\nu_1=0$, then in view of $\int_0^\infty g(t)dt<1$, we obtain that

$$\mathrm{Re}F(\lambda_1)=\sigma_1^2+1+|\xi|^4-\int_0^\infty e^{-\sigma_1 t}g(t)dt>0,$$

it yields contradiction with $(2.3)_1$.

If $\nu_1\neq 0$, then we have that

$$\mathrm{Im}F(\lambda_1)=\nu_1\Big(2\sigma_1+\int_0^\infty\frac{\sin(\nu_1 t)}{\nu_1}e^{-\sigma_1 t}g(t)dt\Big).$$

Next we prove that $\int_0^\infty\frac{\sin(\nu_1 t)}{\nu_1}e^{-\sigma_1 t}g(t)dt>0.$

Denote $a_m=\int_0^{\frac{2m\pi}{|\nu_1|}}\frac{\sin|\nu_1 t|}{|\nu_1|}e^{-\sigma_1 t}g(t)dt$, and we will prove $\{a_m\}_{m=1}^\infty$ is a convergent sequence. By direct computation, we have that

$$a_1=\int_0^{\frac{\pi}{|\nu_1|}}\frac{\sin|\nu_1 t|}{|\nu_1|}\Big(e^{-\sigma_1 t}g(t)-e^{-\sigma_1(t+\frac{\pi}{|\nu_1|})}g(t+\frac{\pi}{|\nu_1|})\Big)dt.$$

Since $\partial_t\big(e^{-\sigma_1 t}g(t)\big)<0$, we have that

$$0<a_1<\int_0^{\frac{\pi}{|\nu_1|}}te^{-\sigma_1 t}g(t)dt.$$

Similarly,

$$a_{m+1}-a_m=\int_{\frac{2m\pi}{|\nu_1|}}^{\frac{2m\pi+\pi}{|\nu_1|}}\frac{\sin|\nu_1 t|}{|\nu_1|}\Big(e^{-\sigma_1 t}g(t)-e^{-\sigma_1(t+\frac{\pi}{|\nu_1|})}g(t+\frac{\pi}{|\nu_1|})\Big)dt,$$

so we have that

$$0<a_{m+1}-a_m<\int_{\frac{2m\pi}{|\nu_1|}}^{\frac{2m\pi+\pi}{|\nu_1|}}te^{-\sigma_1 t}g(t)dt.$$

It yields that

$$0<a_m<\int_0^{\frac{2m\pi}{|\nu_1|}}te^{-\sigma_1 t}g(t)dt\le\frac{g(0)}{(\sigma_1+C_1)^2},$$

so $\{a_m\}_{m=1}^\infty$ is a bounded and monotonic increasing sequence. Since $a_1>0$, $a(\lambda_1):=\lim_{m\to\infty}a_m>0$. Thus we proved that $\int_0^\infty\frac{\sin(\nu_1 t)}{\nu_1}e^{-\sigma_1 t}g(t)dt>0.$ Also, because $\sigma_1>0$ and $\nu_1\neq 0$, it results that $\mathrm{Im}F(\lambda_1)\neq 0$. This contradicts with $(2.3)_2$. Thus by contradiction we proved the claim $\sigma_1\le 0$.

Combining the two cases, we know that $\frac{\lambda}{F(\lambda)}$ is analytic in $\{\lambda\in\mathbb{C};\mathrm{Re}(\lambda)>0\}$. Take $\lambda=\sigma+i\nu$, $\sigma>\max\{\mathrm{Re}\lambda_s\}$, here $\{\lambda_s\}$ is the set of all the singular points of $F(\lambda)$, then by standard calculation we can prove that $\mathrm{L}^{-1}\left[\frac{\lambda}{F(\lambda)}\right](t)$ converges.

The constant C in the expression of $\hat{G}(\xi,t)$ and σ are determined by the initial data of $G(x,t)$. So far we complete the proof. W

In consideration of Lemma 2.1 and Duhamel principle, the solution to the problem (1.1)-(1.2) can be expressed as following：

$$u(t)=G(t)*u_0+H(t)*u_1. \quad (2.4)$$

3. Decay properties of solution operators

In this section, we think of a way to obtain the next decay estimates of the solution operators $G(t)*$ and $H(t)*$ arising in the solution formula (2.4).

Proposition 3.1

Let k and l be integers, $\varphi\in H^{s+1}(R^n)$, $\psi\in H^{s-1}(R^n)$, then the next estimates hold:

1) $$\|\partial_x^k G(t)*\varphi\|_{L^2} \le Ce^{-Ct}\|\varphi\|_{L^2}+C(1+t)^{-\frac{l}{4}}\|\partial_x^{l+k}\varphi\|_{L^2},$$

for $k\ge 0$, $l\ge 0$, $l+k\le s+1$.

2) $$\|\partial_x^k G_t(t)*\varphi\|_{L^2} \le Ce^{-Ct}\|\varphi\|_{L^2}+C(1+t)^{-\frac{l}{4}}\|\partial_x^{l+k+2}\varphi\|_{L^2},$$

for $k\ge 0$, $l\ge 0$, $l+k\le s-1$.

3) $$\|\partial_x^k H(t)*\psi\|_{L^2} \le Ce^{-Ct}\|\psi\|_{L^2}+C(1+t)^{-\frac{l}{4}-\frac{1}{2}}\|\partial_x^{l+k}\psi\|_{L^2},$$

for $k\ge 0$, $l+2\ge 0$, $0\le l+k\le s-1$.

4) $$\|\partial_x^k H_t(t)*\psi\|_{L^2} \le Ce^{-Ct}\|\psi\|_{L^2}+C(1+t)^{-\frac{l}{4}}\|\partial_x^{l+k}\psi\|_{L^2},$$

for $k\ge 0$, $l\ge 0$, $l+k\le s-1$.

To testify Proposition 3.1, the most important step is to gain the point-wise estimates of the fundamental solutions in the Fourier space. In fact we can obtain this by using the following point-wise estimate of solutions to the linear problem (1.1)-(1.2).

Lemma 3.2

Assume u is the solution of (1.1)-(1.2), then it satisfies the following point-wise estimate in the Fourier space:

$$|\hat u_t(\xi,t)|^2+(1+|\xi|^4)|\hat u(\xi,t)|^2+(g\Box\hat u)(\xi,t) \le Ce^{-C\rho(\xi)t}\left(|\hat u_1(\xi)|^2+(1+|\xi|^4)|\hat u_0(\xi)|^2\right),$$

here $\rho(\xi)=\dfrac{1}{1+|\xi|^4}$.

To prove Lemma 3.2, we denote some notations. For any real or complex-valued function $f(t)$, we define

$$(g*f)(t):=\int_0^t g(t-\tau)f(\tau)d\tau,$$

$$(g\Diamond f)(t):=\int_0^t g(t-\tau)(f(\tau)-f(t))d\tau,$$

$$(g\Box f)(t):=\int_0^t g(t-\tau)|f(t)-f(\tau)|^2d\tau.$$

We have the following lemma by direct calculation, which is useful in obtaining our point-wise estimate of solution in the Fourier space.

Lemma 3.2

For any function $k\in C(R)$, and any $\phi\in W^{1,2}(0,T)$, it holds that

1) $(k*\phi)(t)=(k\Diamond\phi)(t)+\int_0^t k(\tau)d\tau\phi(t)$,

2) $$\mathrm{Re}\{(k*\phi)(t)\bar\phi_t(t)\} =-\frac{1}{2}k(t)|\phi(t)|^2+\frac{1}{2}(k'\Box\phi)(t) -\frac{1}{2}\frac{d}{dt}\{(k\Box\phi)(t)-(\int_0^t k(\tau)d\tau)|\phi(t)|^2\},$$

3) $|(k\Diamond\phi)|^2\le(\int_0^t|k(\tau)|d\tau)(|k|\Box\phi)(t)$.

Next we will obtain the point-wise estimates in the Fourier space of solutions to the problem (1.1)-(1.2).

Proof of Lemma 3.2.

Step1: By using Fourier transform to (1.1) we get the following equality:

$$\hat u_{tt}+(1+|\xi|^4)\hat u-g*\hat u=0. \quad (3.2)$$

Multiplying (3.2) by $\bar{\hat u}_t$ we obtain the next equality by taking the real part,

$$Re\left\{\bar{\hat u}_t\left(\hat u_{tt}+(1+|\xi|^4)\hat u-g*\hat u\right)\right\}=0.$$

It yields that

$$\{\frac{1}{2}|\hat u_t|^2+\frac{1}{2}(1+|\xi|^4)|\hat u|^2\}_t-Re\{g*\hat u\bar{\hat u}_t\}=0. \quad (3.3)$$

To the term $Re\{g*\hat u\bar{\hat u}_t\}$ in (3.3) apply 2) in Lemma 3.3 we have that

$$Re\{g*\hat u\bar{\hat u}_t\}=-\frac{1}{2}g(t)|\hat u|^2+\frac{1}{2}(g'\Box\hat u)(t) -\frac{1}{2}\frac{d}{dt}\left\{(g\Box\hat u)(t)-\int_0^t g(\tau)d\tau|\hat u|^2\right\}.$$

We denote

$$E_1(\xi,t)=|\hat u_t|^2+(1+|\xi|^4)|\hat u|^2+g\Box\hat u-(\int_0^t g(s)ds)|\hat u|^2,$$

$$F_1(\xi,t)=g|\hat u|^2-g'\Box\hat u,$$

then we have that

$$\frac{\partial}{\partial t}E_1(\xi,t)+F_1(\xi,t)=0. \quad (3.4)$$

Step 2: Multiplying (3.2) by $\{-(g*\bar{\hat u})_t\}$ and we obtain the next equality by taking the real part,

$$Re\left\{-(g*\bar{\hat{u}})_t\right\}\left\{\hat{u}_{tt}+(1+|\xi|^4)\hat{u}-g*\hat{u}\right\}=0.$$

It results that

$$\{\frac{1}{2}|g*\hat{u}|^2\}_t - Re\{(g*\bar{\hat{u}})_t\hat{u}_{tt}\}$$
$$-Re\{(1+|\xi|^4)\hat{u}(g*\bar{\hat{u}})_t\}=0. \quad (3.5)$$

Due to $(g*\bar{\hat{u}})_t = g(0)\bar{\hat{u}} + g'*\bar{\hat{u}}$, the second term in (3.5) yields that

$$-Re\{\hat{u}_{tt}(g*\bar{\hat{u}})_t\}$$
$$=-Re\{\hat{u}_t(g*\bar{\hat{u}})_t\}_t + Re\{\hat{u}_t(g*\bar{\hat{u}})_{tt}\}$$
$$=-Re\{\hat{u}_t(g*\bar{\hat{u}})_t\}_t + Re\left\{\hat{u}_t\left(g(0)\bar{\hat{u}}_t+(g'*\bar{\hat{u}})_t\right)\right\}$$
$$=-Re\{\hat{u}_t(g*\bar{\hat{u}})_t\}_t + Re\{g(0)|\hat{u}_t|^2+\hat{u}_t(g'*\bar{\hat{u}})_t\}.$$

We denote

$$E_2(\xi,t)=\frac{1}{2}|g*\hat{u}|^2 - Re\{\hat{u}_t(g*\bar{\hat{u}})_t\},$$
$$F_2(\xi,t)=g(0)|\hat{u}_t|^2,$$
$$R_2(\xi,t)=Re\{-\hat{u}_t(g'*\bar{\hat{u}})_t+(1+|\xi|^4)\hat{u}(g*\bar{\hat{u}})_t\},$$

then obtain that

$$\frac{\partial}{\partial t}E_2(\xi,t)+F_2(\xi,t)=R_2(\xi,t). \quad (3.6)$$

Step 3: Multiplying (3.2) by $\bar{\hat{u}}$ and we obtain the next equality by taking the real part,

$$Re\{\bar{\hat{u}}(\hat{u}_{tt}+(1+|\xi|^4)\hat{u}-g*\hat{u})\}=0.$$

It yields that

$$Re\left\{\hat{u}_t\bar{\hat{u}}\right\}_t - |\hat{u}_t|^2 + (1+|\xi|^4)|\hat{u}|^2 -$$
$$Re\left\{g*\hat{u}\bar{\hat{u}}\right\}=0. \quad (3.7)$$

Due to 1) in Lemma 3.3 , we obtain that

$$Re\{g*\hat{u}\bar{\hat{u}}\}=(\int_0^t g(s)ds)|\hat{u}|^2+Re\{g\Diamond\hat{u}\bar{\hat{u}}\}.$$

We denote

$$E_3(\xi,t)=Re\{\hat{u}_t\bar{\hat{u}}\},$$
$$F_3(\xi,t)=(1+|\xi|^4)|\hat{u}|^2-(\int_0^t g(s)ds)|\hat{u}|^2,$$
$$R_3(\xi,t)=|\hat{u}_t|^2+Re\{g\Diamond\hat{u}\bar{\hat{u}}\},$$

then (3.7) yields that

$$\frac{\partial}{\partial t}E_3(\xi,t)+F_3(\xi,t)=R_3(\xi,t). \quad (3.8)$$

Define $\rho(\xi)=\frac{1}{1+|\xi|^4}$, and denote

$$E(\xi,t)=E_1(\xi,t)+\rho(\xi)(\alpha E_2(\xi,t)+\beta E_3(\xi,t)),$$
$$F(\xi,t)=F_1(\xi,t)+\rho(\xi)(\alpha F_2(\xi,t)+\beta F_3(\xi,t)),$$
$$R(\xi,t)=\rho(\xi)(\alpha R_2(\xi,t)+\beta R_3(\xi,t)),$$

where α,β are positive constants, then (3.4), (3.6) and (3.8) yield that

$$\frac{\partial}{\partial t}E(\xi,t)+F(\xi,t)=R(\xi,t). \quad (3.9)$$

We introduce Lyapunov functionals:

$$E_0(\xi,t)=|\hat{u}_t|^2+(1+|\xi|^4)|\hat{u}|^2+g\Box\hat{u},$$
$$F_0(\xi,t)=g\Box\hat{u}+g|\hat{u}|^2.$$

We know that there exist some positive constants c_i (i=1,2,3) from the definitions of $E_1(\xi,t)$ and $F_1(\xi,t)$, such that the following inequalities hold:

$$\begin{cases} c_1E_0(\xi,t)\le E_1(\xi,t)\le c_2E_0(\xi,t), \\ F_1(\xi,t)\ge c_3F_0(\xi,t). \end{cases} \quad (3.10)$$

On the other hand,

$$|E_2(\xi,t)|\le C(|\hat{u}_t|^2+|\hat{u}|^2+g\Box\hat{u}),$$
$$|E_3(\xi,t)|\le C(|\hat{u}_t|^2+|\hat{u}|^2),$$
$$|\rho(\xi)(\alpha E_2(\xi,t)+\beta E_3(\xi,t))|$$
$$\le c_4(\alpha+\beta)E_0(\xi,t).$$

Choose α,β properly small such that

$$c_4(\alpha+\beta)\le min(\frac{c_1}{2},\frac{c_2}{2}),$$

from (3.10) we have that

$$\frac{c_1}{2}E_0(\xi,t)\le E(\xi,t)\le\frac{3c_2}{2}E_0(\xi,t). \quad (3.11)$$

By virtue of (3.10) and noticing that $0\le\int_0^t g(s)ds\le 1$, it is not hard to prove that

$$F(\xi,t)\ge c_3F_0(\xi,t)+$$
$$\rho(\xi)\left\{\alpha g(0)|\hat{u}_t|^2+\frac{\beta}{2}(1+|\xi|^4)|\hat{u}|^2\right\}. \quad (3.12)$$

By virtue of Lemma 3.3, we have that

$$|R_2(\xi,t)|\le\varepsilon|\hat{u}_t|^2+\delta(1+|\xi|^4)|\hat{u}|^2+$$
$$C_{\varepsilon,\delta}(1+|\xi|^4)F_0(\xi,t),$$

and

$$|R_3(\xi,t)|\le|\hat{u}_t|^2+\gamma|\hat{u}|^2+C_\gamma g\Box\hat{u},$$

where $\varepsilon,\delta,\gamma$ are positive constants. Then it is easy to get that

$$|R(\xi,t)|\le\rho(\xi)\{(\alpha\varepsilon+\beta)|\hat{u}_t|^2$$
$$+(\alpha\delta+\beta\gamma)(1+|\xi|^4)|\hat{u}|^2$$
$$+\alpha C_{\varepsilon,\delta}(1+|\xi|^4)F_0(\xi,t)+\beta C_\gamma g\Box\hat{u}\}$$
$$\le(\alpha\varepsilon+\beta)\rho(\xi)|\hat{u}_t|^2+(\alpha\delta+\beta\gamma)|\hat{u}|^2$$
$$+(\alpha+\beta)C_{\varepsilon,\delta,\gamma}F_0(\xi,t).$$

.We claim that there exist $\gamma,\varepsilon,\delta,\alpha,\beta$ such that

$$|R(\xi,t)|\le\frac{1}{2}F(\xi,t). \quad (3.13)$$

First choose $\gamma=\frac{1}{8}$, $\varepsilon=\frac{1}{4}g(0)$, $\delta=\frac{1}{32}g(0)$, $\beta=\frac{1}{4}\alpha g(0)$, then the next three inequalities hold:

$$\frac{1}{4}\beta\geq\alpha\delta+\beta\gamma, \qquad \frac{1}{2}\alpha g(0)\geq\alpha\varepsilon+\beta,$$

$$\frac{1}{2}c_3\geq(\alpha+\beta)C_{\varepsilon,\delta,\gamma}.$$

So as to prove (3.13) (here (3.10) is also considered), it suffices to choose α suitably small such that

$$(1+\frac{1}{4}g(0))\alpha\leq min\{\frac{c_3}{2C_{\varepsilon,\delta,\gamma}},\frac{c_1}{2c_4},\frac{c_2}{2c_4}\}.$$

Due to (3.13) and (3.9), we get that

$$\frac{\partial}{\partial t}E(\xi,t)+\frac{1}{2}F(\xi,t)\leq 0. \quad (3.14)$$

On the other hand, due to (3.11) and (3.12) we obtain that

$$F(\xi,t)>c\rho(\xi)E(\xi,t). \quad (3.15)$$

From (3.14) and (3.15), we have that

$$E(\xi,t)\leq e^{-C\rho(\xi)t}E(\xi,0). \quad (3.16)$$

By virtue of (3.11) and (3.16), we have that

$$|\hat{u}_t|^2+(1+|\xi|^4)|\hat{u}|^2+gW\hat{u}$$
$$\leq Ce^{-C\rho(\xi)t}\left(|\hat{u}_1(\xi)|^2+(1+|\xi|^4)|\hat{u}_0(\xi)|^2\right),$$

so, we obtain the point-wise estimates of solutions to (1.1)-(1.2) in the Fourier space. W

As a simple corollary of Lemma 3.2, we have the following point-wise estimates of the fundamental solutions $G(x,t)$ and $H(x,t)$ in the Fourier space.

Lemma 3.4

$G(x,t)$ and $H(x,t)$ satisfy

1). $|\hat{G}(\xi,t)|\leq Ce^{-C\rho(\xi)t}$;

2). $|\hat{G}_t(\xi,t)|\leq Ce^{-C\rho(\xi)t}(1+|\xi|^4)^{\frac{1}{2}}$;

3). $|\hat{H}(\xi,t)|\leq Ce^{-C\rho(\xi)t}(1+|\xi|^4)^{-\frac{1}{2}}$;

4). $|\hat{H}_t(\xi,t)|\leq Ce^{-C\rho(\xi)t}$,

where $\rho(\xi)=\frac{1}{1+|\xi|^4}$.

Proof.

Putting (2.4) with $u_1=0$ in (3.1), it results that

$$|\hat{G}_t(\xi,t)|^2+(1+|\xi|^4)|\hat{G}(\xi,t)|^2 \leq Ce^{-C\rho(\xi)t}(1+|\xi|^4),$$

it yields 1) and 2).

Putting (2.4) with $u_0=0$ in (3.1), it results that

$$|\hat{H}_t(\xi,t)|^2+(1+|\xi|^4)|\hat{H}(\xi,t)|^2\leq Ce^{-C\rho(\xi)t},$$

it yields 3) and 4). W

Next we use Lemma 3.4 to prove Proposition 3.1

Proof of Proposition 3.1: With a view of 1) in Lemma 3.4 , we have that

$$\|\partial_x^k G(t)*\varphi\|_{L^2}^2 \leq C\int_{R^n}|\xi|^{2k}e^{-C\rho(\xi)t}|\hat{\varphi}(\xi)|^2d\xi$$
$$\leq C\int_{\{\xi:|\xi|\leq1\}}|\xi|^{2k}e^{-\frac{C}{2}t}|\hat{\varphi}|^2d\xi$$
$$+C\int_{\{\xi:|\xi|\geq1\}}|\xi|^{2k}e^{-\frac{Ct}{2|\xi|^4}}|\hat{\varphi}(\xi)|^2d\xi$$
$$\leq Ce^{-Ct}\|\varphi\|_{L^2}^2+C(1+t)^{-\frac{l}{2}}\|\partial_x^{k+l}\varphi\|_{L^2}^2,$$

here $k\geq0$, $l\geq0$, $l+k\leq s+1$. Thus 1) is proved. By virtue of 2), 3) and 4) in Lemma (3.4) , 2), 3) and 4) in Proposition 3.1 can be similarly proved. W

All material on each page should fit within a rectangle of 18 x 23.5 cm (7" x 9.25"), centered on the page, beginning 2.54 cm (1") from the top of the page and ending with 2.54 cm (1") from the bottom. The right and left margins should be 1.9 cm (.75"). The text should be in two 8.45 cm (3.33") columns with a .83 cm (.33") gutter.

4. Decay estimates for linear problem

In this section, we study the decay estimates of solutions of the linear problem (1.1)-(1.2).

Theorem 4.1.

Let $s\geq1$ be an integer. Suppose that $u_0\in H^{s+1}(R^n)$ and $u_1\in H^{s-1}(R^n)$, and set $I_0:=\|u_0\|_{H^{s+1}}+\|u_1\|_{H^{s-1}}$. Then the solution u of the problem (1.1)-(1.2) given by (2.4) satisfies that

$$u\in C^0([0,\infty);H^{s+1}(R^n))\cap C^1([0,\infty);H^{s-1}(R^n)),$$

and the following energy estimate:

$$\|u_t(t)\|_{H^{s-1}}^2+\|u(t)\|_{H^{s+1}}^2$$
$$+\int_0^t(\|u_t(\tau)\|_{H^{s-3}}^2+\|u(\tau)\|_{H^{s-1}}^2)d\tau\leq CI_0^2.$$

Proof. We have obtained the solution u of (1.1)-(1.2) given by (2.4) and proved that it satisfies the point-wise estimates (3.1) in the Fourier space. Due to (3.14) and (3.15) we obtain that $\frac{\partial}{\partial t}E(\xi,t)+C\rho(\xi)E(\xi,t)\leq0.$ Integrate the above inequality with respect to t and use the inequality (3.11), so we have that

$$E_0(\xi,t)+\int_0^t\rho(\xi,t)E_0(\xi,\tau)d\tau\leq CE_0(\xi,0). \quad (4.1)$$

Multiply (4.1) by $(1+|\xi|^2)^{s-1}$ and integrate the resulting inequality with respect to $\xi\in R^n$, then we obtain that

$$\int_{R^n}(1+|\xi|^2)^{s-1}E_0(\xi,t)d\xi$$
$$+\int_{R^n}(1+|\xi|^2)^{s-1}\int_0^t\rho(\xi,t)E_0(\xi,\tau)d\tau d\xi$$
$$\le C\int_{R^n}(1+|\xi|^2)^{s-1}E_0(\xi,0)d\xi,$$

it yields that

$$\|u_t(t)\|^2_{H^{s-1}}+\|u(t)\|^2_{H^{s+1}}$$
$$+\int_0^t(\|u_t(\tau)\|^2_{H^{s-3}}+\|u(\tau)\|^2_{H^{s-1}})d\tau\le CI_0^2. \quad (4.2)$$

(4.2) guarantees the regularity of the solution (2.4). So far we complete the proof of Theorem 4.1. $\blacksquare$

By using Proposition 3.1 we obtain the following decay estimates of u given by (2.4), if initial data $u_0\in H^{s+1}(R^n)$ and $u_1\in H^{s-1}(R^n)$.

Theorem 4.2.

With the same conditions as Theorem 4.1, if $d\in Z_+$, then u given by (2.4) satisfies the following decay estimate:

$$\|\partial_x^k u(t)\|_{H^{s+1-k-d}}\le CI_0(1+t)^{-\frac{d}{4}}, \quad (4.3)$$

here $k\ge 0$, $d\ge 0$, $k+d\le s+1$;

if $d\notin Z_+$, the following decay estimate holds:

$$\|\partial_x^k u(t)\|_{H^{s-k-[d]}}\le CI_0(1+t)^{-\frac{d}{4}}, \quad (4.4)$$

for $k\ge 0$, $d\ge 0$, $k+[d]\le s$.

Proof.

Assume $k\ge 0$, $m\ge 0$ are integers. By using (2.4) and applying 1) and 3) in Proposition 3.1, we have that

$$\|\partial_x^{k+m}u(t)\|_{L^2}$$
$$\le\|\partial_x^{k+m}G(t)*u_0\|_{L^2}+\|\partial_x^{k+m}H(t)*u_1\|_{L^2}$$
$$\le Ce^{-C(t+1)}\|u_0\|_{L^2}+C(1+t)^{-\frac{l_1}{4}}\|\partial_x^{k+m+l_1}u_0\|_{L^2}$$
$$+Ce^{-C(t+1)}\|u_1\|_{L^2}+C(t+1)^{-\frac{l_2}{4}-\frac{1}{2}}\|\partial_x^{k+m+l_2}u_1\|_{L^2}$$
$$\le Ce^{-C(1+t)}\|(u_0,u_1)\|_{L^2}$$
$$+C(1+t)^{-\frac{l_1}{4}}\|\partial_x^{k+m+l_1}u_0\|_{L^2}$$
$$+C(1+t)^{-\frac{l_2}{4}-\frac{1}{2}}\|\partial_x^{k+m+l_2}u_1\|_{L^2}, \quad (4.5)$$

here $l_1\ge 0$, $l_2\ge -2$, $k+m+l_1\le s+1$, $0\le k+m+l_2\le s-1$.

Choose the minimal integers l_1 and l_2 satisfying

$$\frac{l_1}{4}\ge\frac{d}{4},\ \frac{l_2}{4}+\frac{1}{2}\ge\frac{d}{4},$$

i.e.

$$l_1=\begin{cases}d, & d\in Z_+;\\ [d]+1, & d\notin Z_+,\end{cases}\quad l_2=l_1-2.$$

At the same time, the next inequality holds:

$$e^{-C(1+t)}\le C(1+t)^{-\frac{d}{4}}.$$

So if $d\in Z_+$, we obtain from (4.5) that

$$\|\partial_x^{k+m}u(t)\|_{L^2}\le CI_0(1+t)^{-\frac{d}{4}},$$

for $0\le m\le s+1-k-d$. Take sum with $0\le m\le s+1-k-d$, we get (4.3).

If $d\notin Z_+$, we obtain from (4.5) that

$$\|\partial_x^{k+m}u(t)\|_{L^2}\le CI_0(1+t)^{-\frac{d}{4}},$$

for $0\le m\le s-k-[d]$. Take sum with $0\le m\le s-k-[d]$, then we get (4.4). $\blacksquare$

Remark 1. With the same conditions as Theorem 4.1, through the similar proof to Theorem 4.2 we have the following estimates:

if $d\in Z_+$, $\|\partial_x^k u_t(t)\|_{H^{s-1-k-d}}\le CI_0(1+t)^{-\frac{d}{4}}$,

for $k\ge 0$, $d\ge 0$, $k+d\le s-1$;

if $d\notin Z_+$, $\|\partial_x^k u_t(t)\|_{H^{s-2-k-[d]}}\le CI_0(1+t)^{-\frac{d}{4}}$,

for $k\ge 0$, $d\ge 0$, $k+[d]\le s-2$.

Remark 2. In the special case $d=k$ in Theorem 4.2, we obtain the following estimate:

$$\|\partial_x^k u(t)\|_{H^{s+1-2k}}\le CI_0(1+t)^{-\frac{k}{4}},$$

here $k\ge 0$, $2k\le s+1$.

Remark 3. The estimates in Theorem 4.1 and Theorem 4.2 indicate that the decay structure of solutions to the linear problem (4.1)-(4.2) is of regularity-loss type. To have $\frac{d}{4}$-order decay, we have to lose d-order regularity.

Remark 4. The condition c) in Assumption [A] plays a key role to obtain the dissipative structure in this paper. If c) is weakened to $1-\int_0^t g(s)ds\ge 0$, then the dissipative structure would be totally different. Take memory kernels $g_1(t)=a\mu e^{-\mu t}$ and $g_2(t)=\mu e^{-\mu t}$ ($a<1$ and μ are constants) as examples, by direct calculation we can see the difference between the two dissipative structures, which in some way reflects the optimality of the dissipative structure in this paper.

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